

Lectures on Thermodynamics

1 Thermodynamics

Thermodynamics studies the macroscopic behavior of the many-body systems and describes them in terms of macroscopic parameters characterizing the system which are **temperature** T , **pressure** P and **volume** V . Equation of state is the relation among these parameters and is written as

$$f(P, T, V) = 0$$

The basic assumption is that such an equation **always exists**. Different functions describing thermodynamical systems are divided in two groups:

- functions of **state** which are complete differentials and depend only on initial and final state.

$$\oint df(x) = 0$$

$$\int_1^2 df(x) = f(2) - f(1)$$

Examples are thermodynamical potentials (see later).

- functions of **process** which are not complete differentials and depend on the **path** taken between initial and final state.

$$\oint d'f(x) \neq 0$$

Examples are work and heat.

Let us look at the work done during an isothermal $T = \text{const.}$ and an isobaric $P = \text{const.}$ process. The work is

$$A'_T = \int_{V_1}^{V_2} \frac{nRT}{V} dV = nRT \ln \frac{V_2}{V_1}$$

$$A'_P = \int_{V_1}^{V_2} P dV = P(V_2 - V_1)$$

$$A'_T \neq A'_P$$

Thermodynamical potentials are basic functions of state expressed in terms of conjugate coordinates P, V and T, S . Transformation from one coordinate to another is known as **Laplace transform**. They are:

- **Internal energy** $U(S, V)$

$$dU = d'Q - dA' = T dS - P dV$$

$$dU(S, V) = \left(\frac{\partial U}{\partial S} \right)_V dS + \left(\frac{\partial U}{\partial V} \right)_S dV$$

$$T = \left(\frac{\partial U}{\partial S} \right)_V$$

$$P = - \left(\frac{\partial U}{\partial V} \right)_S$$

During an **adiabatic** $\Delta Q = 0$ process

$$A|_Q = -\Delta U = U_i - U_f$$

which means that the work done corresponds to the change of internal energy. In an isochoric $V = \text{const.}$ one finds

$$dU_V \equiv dQ_V = c_V dT$$

$$c_V \equiv \left(\frac{\partial U}{\partial T} \right)_V = \left(\frac{\partial S}{\partial T} \right)_V \left(\frac{\partial U}{\partial S} \right)_V$$

$$c_V = T \left(\frac{\partial S}{\partial T} \right)_V$$

- **Enthalpy** $H(S, P)$ is Laplace transform of $U(S, V)$ in conjugate coordinates $V \rightarrow P$

$$H = U + PV = F + TS + PV$$

$$dH = T dS + V dP$$

$$T = \left(\frac{\partial H}{\partial S} \right)_P$$

$$V = \left(\frac{\partial H}{\partial P} \right)_S$$

During an **isobaric** $P = \text{const.}$ process

$$Q|_P = \Delta H|_P = H_f - H_i$$

which means that the exchange of heat corresponds to the change of enthalpy and

$$dH_P \equiv dQ_P = c_P dT$$

$$c_P = \left(\frac{\partial H}{\partial T} \right)_P = T \left(\frac{\partial S}{\partial T} \right)_P$$

Thus, in an isobaric process, enthalpy plays the same role as internal energy in an isochoric process.

- **Helmoltz free energy** $F(T, V)$ is Laplace transform of $U(S, V)$ in conjugate coordinates $S \rightarrow T$

$$\begin{aligned}
 F &= U - T S \\
 dF &= -S dT - P dV \\
 S &= - \left(\frac{\partial F}{\partial T} \right)_V \\
 P &= - \left(\frac{\partial F}{\partial V} \right)_T \\
 \left(\frac{\partial S}{\partial V} \right)_T &= - \left(\frac{\partial^2 F}{\partial T \partial V} \right)_{T,V} \\
 \left(\frac{\partial P}{\partial T} \right)_V &= - \left(\frac{\partial^2 F}{\partial T \partial V} \right)_{T,V} \\
 \left(\frac{\partial P}{\partial T} \right)_V &= \left(\frac{\partial S}{\partial V} \right)_T
 \end{aligned}$$

in an reversible **isothermal** process $dF = -P dV$ so that

$$A_T = F_i - F_f$$

and the work extracted from the system is equal to the loss of free energy. In case of **irreversible** process one has

$$A_T \leq F_i - F_f$$

- **Gibbs free energy** $G(T, P)$ is Laplace transform of $F(T, V)$ in conjugate coordinates $V \rightarrow P$

$$\begin{aligned}
 G &= F + P V \\
 dG &= -S dT + V dP \\
 S &= - \left(\frac{\partial G}{\partial T} \right)_P \\
 V &= \left(\frac{\partial G}{\partial P} \right)_T
 \end{aligned}$$

Under constant T and P the Gibbs function satisfies

$$\frac{dG}{dt} \leq 0$$

and is decreasing in time reaching its minimum at the thermodynamic equilibrium.

1.1 Equation of state of a perfect gas

Perfect or ideal gas is a low pressure approximation of a real gas with the following characteristics:

1. molecules are point-like structures
2. no interaction among particles of gas

Equation of state of a real gas can be obtained combining empirical Boyle-Mariott and Guy-Lisac laws

$$\begin{aligned}
 PV &= \text{const.} \quad T = \text{const.} \\
 P &= P_0 (1 + \alpha_P \Delta t) \quad V = \text{const.} \\
 V &= V_0 (1 + \alpha_V \Delta t) \quad P = \text{const.} \\
 \alpha_P &= \alpha_V = \frac{1}{273 \text{ K}} \\
 P &= \frac{P_0}{T_0} (T_0 + T_0 \alpha_P (T - T_0)) = \frac{P_0}{T_0} T \\
 \frac{P}{T} &= \frac{P_0}{T_0} \\
 \frac{V}{T} &= \frac{V_0}{T_0}
 \end{aligned}$$

Let us take a gas from the state P_1, V_1, T_1 to a state P_1, V_2', T_2 . Guy-Lisac gives

$$V_2' = \frac{T_2}{T_1} V_1$$

now we take it from P_1, V_2', T_2 to the final state P_2, V_2, T_2 . Boyle-Marott gives

$$P_1 V_2' = P_2 V_2$$

combining together the two results one finds

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

leading to the perfect gas law

$$\boxed{P \cdot V = N k T} \tag{1}$$

where N is the number of molecules and $k = 1,38 \cdot 10^{-23} \text{ J/K}$ is the Boltzman constant. It is much easier to calculate the number of **moles** than a number of molecules for a gas so it is convenient to rewrite the gas equation as follows

$$P \cdot V = N k T = n N_A k T$$

$$R = N_A k$$

$$P \cdot V = n R T$$

where n is number of moles, $N_A = 6,02 \cdot 10^{23}$ Avogadro's number and
 $R = N_A k = 8,3 \cdot J/K$ is gas constant.

- Internal energy of perfect gas does not depend on volume (Joule's law) so

$$dU = \left(\frac{\partial U}{\partial T} \right)_V dT + \left(\frac{\partial U}{\partial V} \right)_T dV = c_V dT$$

$$U = U_0 + c_V T$$

- Enthalpy

$$H = U + PV = U_0 + c_V T + nRT$$

$$H = c_P T + U_0$$

- The entropy of ideal gas is

$$dS(T, V) = \frac{\partial S}{\partial T} dT + \frac{\partial S}{\partial V} dV$$

$$dS(T, V) = c_V \frac{dT}{T} + \frac{\partial P}{\partial T} dV$$

$$dS(T, V) = c_V \frac{dT}{T} + \frac{nR}{V} dV$$

$$S(T, V) = S_0 + c_V \ln \frac{T}{T_0} + nR \ln \frac{V}{V_0}$$

$$S(T, P) = S_0 + (c_P - nR) \ln \frac{T}{T_0} + nR \ln \frac{V}{V_0} = S_0 + c_P \ln \frac{T}{T_0} - nR \ln \frac{P}{P_0}$$

$$S(V, P) = S_0 + c_P \ln \frac{V}{V_0} + c_V \ln \frac{P}{P_0}$$

- Helmholtz free energy

$$F = U - TS = U_0 + c_V T - T \left(S_0 + c_V \ln \frac{T}{T_0} + nR \ln \frac{V}{V_0} \right)$$

$$F = c_V T \left(1 - \ln \frac{T}{T_0} \right) - nR \ln \frac{V}{V_0} - S_0 T + U_0$$

- Gibbs free energy

$$G = U - TS + PV = U_0 + c_V T - T \left(S_0 + c_P \ln \frac{T}{T_0} + nR \ln \frac{P}{P_0} \right) + nRT$$

$$G = c_P T \left(1 - \ln \frac{T}{T_0} \right) - nR \ln \frac{P}{P_0} - S_0 T + U_0$$

1.2 PV diagrams

These diagrams describe work performed

$$A = \int P dV$$

$$A|_T = \int P dV = nRT \int_{V_1}^{V_2} dV/V = nRT \ln \frac{V_2}{V_1}$$

$$A|_P = \int P dV = P(V_2 - V_1)$$

$$A|_V = 0$$

1.3 TS diagrams

It is usual to define $\gamma = c_P/c_V$, $c_V = nR/\gamma - 1$, $c_P = nR\gamma/\gamma - 1$ and one can write

$$S - S_0 = (c_P - c_V) \ln \frac{V}{V_0} + c_V \ln \frac{T}{T_0}$$

$$S - S_0 = \ln \left(\frac{V}{V_0} \right)^{c_P} \left(\frac{P}{P_0} \right)^{c_V} = \ln \left(\frac{V}{V_0} \right)^{nR\gamma/\gamma-1} \left(\frac{P}{P_0} \right)^{nR/\gamma-1}$$

$$S - S_0 = \frac{nR}{\gamma-1} \ln \left(\frac{V}{V_0} \right)^\gamma \left(\frac{P}{P_0} \right)$$

$$PV^\gamma = \text{const.} e^{(\gamma-1)S/nR}$$

$$TV^{\gamma-1} = \text{const.} e^{(\gamma-1)S/nR}$$

$$TP^{\gamma-1/\gamma} = \text{const.} e^{(\gamma-1)S/nR\gamma}$$

thus one obtains T, S diagrams describing the exchange of **heat** as

$$\Delta Q = \int_{S_1}^{S_2} T(S) dS$$

$$T(S)|_V = \text{const.} e^{(\gamma-1)S/nR}$$

$$T(S)|_P = \text{const.} e^{(\gamma-1)S/nR\gamma}$$

1.4 Adiabatic processes

A thermodynamical transformation is called **adiabatic** if no heat is exchanged during the process $dQ = 0$. A real transformation taking place very quickly is a good approximation of an adiabatic process. The work is done on the expenses of internal energy

$$0 = c_V dT + P dV$$

$$\frac{dV}{V} = -\frac{c_V}{nR} \frac{dT}{T}$$

$$0 = d \ln \left(\frac{T}{T_0} \right) \left(\frac{V}{V_0} \right)^{\gamma-1}$$

$$T V^{\gamma-1} = \text{const.}$$

$$P V^\gamma = \text{const.}$$

$$T P^{(1-\gamma)/\gamma} = \text{const.}$$

The work in an adiabatic process is

$$A = \text{const.} \int_{V_1}^{V_2} dV/V^\gamma = \frac{\text{const.}}{1-\gamma} V_1^{1-\gamma} \left[1 - \left(\frac{V_1}{V_2} \right)^{\gamma-1} \right] \quad (2)$$

$$\text{const.} V_1^{\gamma-1} = P_1 V_1 = nRT_1 \quad (3)$$

$$A = \frac{nRT_1}{\gamma-1} \left(1 - \frac{T_2}{T_1} \right) \quad (4)$$

1.5 Polytropic equation

This is a process during which the **heat** capacity, defined in general as $dQ = c dT$, remains constant. From the first law and ideal gas law one has

$$c dT = c_V dT + P dV$$

$$P dV + V dP = R dT$$

$$P dV + V dP = R \frac{P dV}{c - c_V}$$

$$(c - c_P) \frac{dV}{V} = (c - c_V) \frac{dP}{P}$$

$$P V^n = \text{const.}$$

$$n = \frac{c - c_P}{c - c_V}$$

$$c = \frac{n c_V - c_P}{n - 1}$$

n is known as polytropic exponent. One can have the following situations

- $n = 0$ isobaric process $c = c_P$

$$P = \text{const.}$$

- $n = \infty$ isochoric process $c = c_V$

$$V = \text{const.}$$

- $n = 1$ isothermal process $c = \infty$

$$T = \text{const.}$$

$n = c_P/c_V$ adiabatic process $c = 0$

$$dQ = 0$$

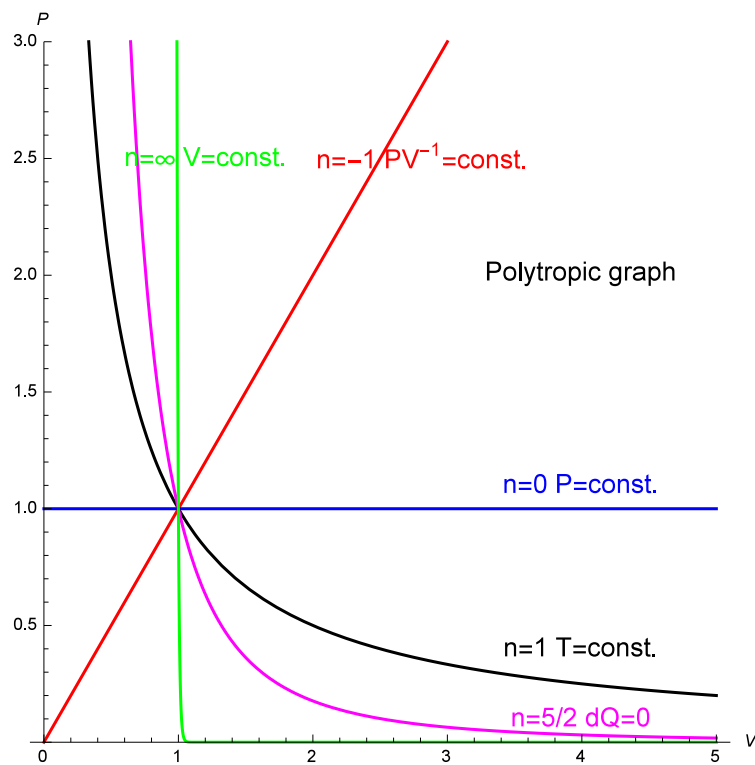


Figure 1: PV graph of a polytropic process

while T, S graph is obtained from

$$T = \text{const.} \cdot e^{(n-1)S/nc_V - c_P} = \text{const.} \cdot e^{S/c}$$

2 Van der Waals equation for real gas

Real gas has **interacting** molecules of **finite** size and these facts have to be taken into account. First of all the real volume **available** to the molecules is less than V

due to the volume of interaction of each molecule b . This volume corresponds to the half (facing incoming molecule) of **cross section** of two molecules given by

$$b = \frac{1}{2}N\frac{4\pi}{3}(2R)^2 = \frac{16}{3}N\pi R^3$$

so one finds

$$P(V - b) = nRT$$

this is known as Clausius equation. Now take into account intermolecular interactions of order $1/r^6$ (Van der Waals forces) which **reduce** pressure of gas due to the density of molecules in the last two layers of gas. This net force decreases the force exerted onto the wall by the particles in the surface layer. The net force on a surface particle, pulling it into the container, is proportional to the number density

$$C = N/V$$

The number of particles in the surface layers is, again by assuming homogeneity, also proportional to the density. In total, the force on the walls is decreased by a factor proportional to the square of the density, and the pressure (force per unit surface) is decreased by

$$aC^2 = a\left(\frac{N}{V}\right)^2$$

Thus, modified equation is

$$\left[P + a\left(\frac{N}{V}\right)^2 \right] (V - b) = nRT$$

if we introduce abbreviations as $a = a N_A^2, b = b N_A/N$ the equation can be written as

$$RT = \left(P + \frac{a}{V^2} \right) (V - b)$$

$$P = \frac{RT}{V - b} - \frac{a}{V^2}$$

which is Van der Waals equation for one mole. For large volume $V \rightarrow \infty$ it becomes the equation of perfect gas. VDW equation is **cubic** equation having an inflexion point characterized by **critical** values of T_c, P_c, V_c . Critical values correspond to

coinciding roots of the cubic equation $V_1 = V_2 = V_3$ that can be found as follows.

$$\begin{aligned}
 0 &= (V - V_c)^3 \\
 0 &= V^3 - 3V^2 V_c + 3V V_c^2 - V_c^3 \\
 0 &= V^3 - \left(\frac{RT}{P} + b\right) V^2 + V \frac{a}{P} - \frac{ab}{P} \\
 3V_c &= \frac{RT_c}{P_c} + b \\
 3V_c^2 &= \frac{a}{P_c} \\
 V_c^3 &= \frac{ab}{P_c} \\
 T_c &= \frac{8a}{27bR} \\
 P_c &= \frac{a}{27b^2} \\
 V_c &= 3b
 \end{aligned}$$

VDW equation can be written in **reduced** form which is independent of type of gas involved. Introduce notation $t = T/T_c, p = P/P_c, v = V/V_c$ to find

$$p = \frac{8t}{3v-1} - \frac{3}{v^2}$$

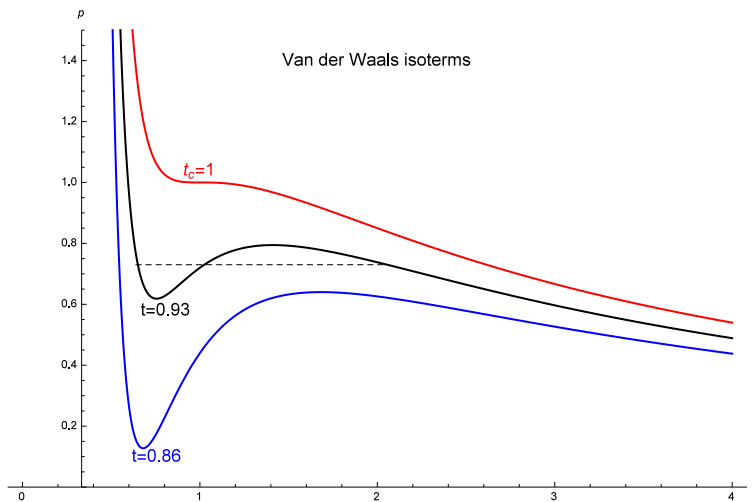


Figure 2: *PV* graph of Van der Waals gas

3 Statics and dynamics of a rigid body

Rigid body is defined as a collection of particles (atoms and molecules) that maintain fixed position with respect to each other. Therefore, a rigid body moves as a whole, rigid object. To every rigid body we associate a point called **centre of mass**, where we can consider concentrated the total mass of the object. In other words, the extended object can be replaced by its centre of mass. The motion of a rigid body can thus be split into a motion of its centre of mass and a motion **around** an axis passing through its centre of mass. The centre of mass is defined as

$$\vec{r}_{CM} = \frac{\sum_n m_n \vec{r}_n}{M_{tot}}$$

$$\vec{r}_{CM} = \frac{1}{M} \int \vec{r} \rho dV$$

Example: The position of centre of mass of a circular section of angle φ_0 .

$$\vec{r}_{CM} = \frac{1}{M} \int \vec{r} \rho dV$$

$$\vec{r}_{CM} = \frac{\rho}{M} \int_0^R r^2 dr \int_0^{\varphi_0} (\cos \varphi \vec{i} + \sin \varphi \vec{j}) d\varphi$$

$$\vec{r}_{CM} = \rho \frac{R^3}{3M} (\sin \varphi_0 \vec{i} + (1 - \cos \varphi_0) \vec{j})$$

$$M = \rho \int_0^R r dr \int_0^{\varphi_0} d\varphi = \rho \frac{R^2}{2} \varphi_0$$

$$\vec{r}_{CM} = \frac{2R}{3\varphi_0} (\sin \varphi_0 \vec{i} + (1 - \cos \varphi_0) \vec{j})$$

We can consider translation and rotation of a rigid body. Physical quantities characterizing translation are linear momentum $\vec{p} = m\vec{v}$ and force $\vec{F} = d\vec{p}/dt$.

Rotation is characterized by **angular momentum** $\vec{L} = \vec{r} \times \vec{p}$ and by the momentum of force or twisting force known as **torque** (Latin torquere=to twist) $\vec{M} \equiv \vec{r} \times \vec{F} = d\vec{L}/dt$. The table below put in relation various quantities describing translation and rotation

Translation	Rotation
m -mass	I -moment of inertia
$\vec{p} = m\vec{v}$ -linear momentum	$\vec{L} = I\vec{\omega}$ -angular momentum
$\vec{F} = d\vec{p}/dt$ -force	$\vec{M} = d\vec{L}/dt$ -torque
$E_K = mv^2/2$ -kinetic energy	$E_K = I\omega^2/2$ -kinetic energy

One can see that the quantities describing rotation do not depend **only** on mass but also on the distance from the axis of rotation.

The moment of inertia of a rigid body is defined as

$$I = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta m_i \bar{r}_i^2 = \int \bar{r}^2 \rho dV$$

where ρ is the volume density of the substance. The angular momentum is

Examples:

1. The moment of inertia of a circular section of angle φ_0 .

$$\begin{aligned} I &= \int \bar{r}^2 \rho dV \\ I &= \rho \int_0^R r^3 dr \int_0^{\varphi_0} d\varphi = \rho \varphi_0 \frac{R^4}{4} \\ M &= \rho \pi R^2 \\ M_\varphi &= \rho R^2 \frac{\varphi_0}{2} = \frac{\varphi_0}{2\pi} M \\ I_{\varphi_0} &= \frac{\varphi_0}{4\pi} M R^2 \end{aligned}$$

2. The moment of inertia of a circular ring of width $R_2 - R_1$.

$$\begin{aligned} I &= I_{R_2} - I_{R_1} = \frac{1}{2} (M_2 R_2^2 - M_1 R_1^2) \\ M_1 &= \rho \pi R_1^2 \\ M_2 &= \rho \pi R_2^2 \\ M &= M_2 - M_1 = \frac{M_1}{R_1^2} (R_2^2 - R_1^2) \\ I &= \frac{1}{2} \rho \pi (R_2^4 - R_1^4) \\ I &= \frac{1}{2} M (R_2^2 + R_1^2) \end{aligned}$$

3. Moment of inertia of a cylinder of radius R and height h around an axis z passing through its centre

$$\begin{aligned} I_z &= \rho \int r^2 dV = \rho \int_0^R r^3 dr \int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} dz \\ I_z &= \rho \frac{\pi}{2} R^4 h \\ M &= \rho \pi R^2 h \\ I_z &= \frac{1}{2} M R^2 \end{aligned}$$

4. Moment of inertia of a cylinder of radius R and height h around an axis x passing through the centre of its height h

$$I_x = \rho \int (z^2 + y^2) dV = \rho \int_0^R r dr \int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} (r^2 \sin^2 \varphi + z^2) dz$$

$$I_x = \rho \int_0^R r dr \int_{-h/2}^{h/2} dz (r^2 \pi + 2\pi z^2)$$

$$I_x = \rho \int_{-h/2}^{h/2} dz \left(\frac{\pi}{4} R^4 + \pi z^2 R^2 \right) = \rho \frac{\pi}{4} R^2 h \left(R^2 + \frac{h^2}{3} \right)$$

$$M = \rho \pi R^2 h$$

$$I_x = \frac{1}{4} M \left(R^2 + \frac{h^2}{3} \right)$$

To show the advantage of choosing a proper basis for calculation let us calculate the area of a circle in Cartesian and polar coordinates.

$R^2 = x^2 + y^2$ circle in Cartesian coordinates

$$A = \int dx dy = \int_{-R/2}^{R/2} dx \int_{-\sqrt{R^2-x^2}/2}^{\sqrt{R^2-x^2}/2} dy = 2 \int_{-R/2}^{R/2} \sqrt{R^2 - x^2} dx$$

$$A = |x = R \sin \alpha| = 2 R \int_{-\pi/2}^{\pi/2} \sin^2 \alpha d\alpha = R^2 \left(\alpha + \frac{\sin 2\alpha}{2} \right) \Big|_{-\pi/2}^{\pi/2}$$

$$A = \pi R^2$$

$R = \text{const.}$ circle in polar coordinates

$$A = \int_0^R r dr \int_0^{2\pi} d\varphi = \pi R^2$$

General form of moment of inertia following from the definition of angular momentum is

$$\vec{L} = m \vec{r} \times \vec{v}_t = m \vec{r} \times (\vec{\omega} \times \vec{r}) = m \sum_{i=1}^3 (x_i^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r}) \vec{x}_i)$$

$$\vec{L} = m \left[\vec{\omega} (x^2 + y^2 + z^2) - (\omega_x x + \omega_y y + \omega_z z) \vec{x}_i \right]$$

$$\vec{L} = m \left[\omega_x (y^2 + z^2) \vec{i} + \omega_y (x^2 + z^2) \vec{j} + \omega_x (y^2 + x^2) \vec{k} \right]$$

$$- (\omega_y y + \omega_z z) x \vec{i} - (\omega_x x + \omega_z z) y \vec{j} - (\omega_y y + \omega_x x) z \vec{k} \Big]$$

$$L_x = m (y^2 + z^2) \omega_x - m x y \omega_y - m x z \omega_z = I_{xx} \omega_x - I_{xy} \omega_y - I_{xz} \omega_z$$

$$L_y = m (x^2 + z^2) \omega_y - m y x \omega_x - m y z \omega_z = I_{yy} \omega_y - I_{yx} \omega_x - I_{yz} \omega_z$$

$$L_z = m (y^2 + x^2) \omega_z - m z y \omega_y - m z x \omega_x = I_{zz} \omega_z - I_{zy} \omega_y - I_{zx} \omega_x$$

One can see that the moments of inertia are components of a second-rank tensor
(matrix)

$$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Fortunately, it is always possible to choose three axis in a way that the **mixed** momenta are zero. These axis are known as **principal axis** and coincide with axis of symmetry of the rigid body. The torque is

$$\begin{aligned} \vec{M} &= \frac{d\vec{L}}{dt} = \sum_i I_{ii} \left(\frac{d\omega_i}{dt} \vec{e}_i + \frac{d\vec{e}_i}{dt} \omega_i \right) \\ \frac{d\vec{e}_i}{dt} &= \vec{\omega} \times \vec{e}_i \\ \vec{M} &= I \cdot \vec{\alpha} + \vec{\omega} \times \vec{L} \rightarrow \text{Euler equations} \end{aligned}$$

in the principal axis system one has

$$\begin{aligned} M_x &= I_x \dot{\omega}_x + (I_z - I_y) \omega_y \omega_z \\ M_y &= I_y \dot{\omega}_y + (I_x - I_z) \omega_z \omega_x \\ M_z &= I_z \dot{\omega}_z + (I_y - I_x) \omega_x \omega_y \end{aligned}$$

To define equilibrium conditions for translation and rotation one requires

$$\begin{aligned} \vec{F}_{tot} &\equiv \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n = 0 \\ \vec{M}_{tot} &\equiv \vec{M}_1 + \vec{M}_2 + \dots + \vec{M}_n = 0 \end{aligned}$$

The above condition lead also to the **conservation** of linear and angular momentum, which, together with conservation of energy represent fundamental conservation laws of physics

$$\begin{aligned} \vec{F}_{tot} = 0 &\Rightarrow \vec{p}_{tot} = \text{const..} \\ \vec{M}_{tot} = 0 &\Rightarrow \vec{L}_{tot} = \text{const..} \end{aligned}$$

Torque can be written in terms of of the components of force \vec{F} and its arm \vec{r} in the Cartesian coordinate system as

$$\begin{aligned} \vec{M} &= \vec{r} \times \vec{F} \\ M_x &= y F_z - z F_y \\ M_y &= z F_x - x F_z \\ M_z &= x F_y - y F_x \end{aligned}$$

Assume that we have two forces \vec{F}_1, \vec{F}_2 , in the y, z plane, producing two torques \vec{M}_1, \vec{M}_2 around a centre of rotation O having an arm \vec{r} . The resulting (total) torque is

$$\begin{aligned}\vec{M}_R &= \vec{M}_1 + \vec{M}_2 \\ M_{x,R} &= y F_{z,1} - z F_{y,1} + y F_{z,2} - z F_{y,2} \\ M_{x,R} &= y (F_{z,1} + F_{z,2}) - z (F_{y,1} + F_{y,2}) \\ M_{x,R} &= y F_{R,z} - z F_{R,y} = M_{x,1} + M_{x,2}\end{aligned}$$

this simple property of vectors goes under the name of Varignon “theorem” in civil engineering. The proof is based on equivalence of areas which simply means that the geometrical meaning of a cross-product is an area formed by the two vectors in the product.